

The Quantum Closet

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Abstract

The equivalence postulate approach to quantum mechanics entails a derivation of quantum mechanics from a fundamental geometrical principle. Underlying the formalism there exists a basic cocycle condition, which is invariant under D -dimensional finite Möbius transformations. The invariance of the cocycle condition under finite Möbius transformations implies that space is compact. Additionally, it implies energy quantisation and the undefinability of quantum trajectories. I argue that the decompactification limit coincides with the classical limit. Evidence for the compactness of the universe may exist in the Cosmic Microwave Background Radiation.

1 Introduction

The synthesis of quantum mechanics and general relativity continues to pose an important challenge in the basic understanding of physics. While quantum mechanics accounts with astonishing success for physical observations at the smallest distance scales, general relativity accomplishes a similar feat at the largest. Yet these two mathematical modellings of the observed data are mutually incompatible. This is seen most clearly in relation to the vacuum. The first predicts a value that is off by orders of magnitude from the observed value, which is determined by using the second. To date there is no solution to this problem. In view of this calamity it seems prudent to explore the foundations of each of these theories, and the fundamental principles that underly them. General relativity follows from a basic geometrical principle, the equivalence principle, whereas the basic tenant of quantum mechanics is the probability interpretation of the wave function.

The question arises whether quantum mechanics can follow from a basic geometrical principle, akin to the geometrical principle that underlies relativity. Starting in [1] we embarked on a rigorous derivation of quantum mechanics from a geometrical principle. The equivalence postulate of quantum mechanics hypothesises that any two physical states can be connected by a coordinate transformation. This includes states which arise under different potentials. In particular, any state may be transformed so as to correspond to that of a free particle at rest. This bears close resemblance to Einstein's equivalence principle that underlies general relativity with an important caveat. While in the case of Einstein's equivalence principle it is the gravitational field which is "locally balanced" by a coordinate transformation, in the equivalence postulate approach to quantum mechanics it is an arbitrary external potential which is "globally balanced" by a coordinate transformation. The equivalence postulate of quantum mechanics is naturally formulated in the framework of Hamilton–Jacobi theory.

The implementation of the equivalence postulate in the context of the Hamilton–Jacobi theory yields a Quantum Hamilton–Jacobi equation. The Classical Hamilton–Jacobi Equation is obtained by requiring the existence of a canonical transformation from one set of phase space variables to a second set of phase space variable such that the Hamiltonian is mapped to a trivial Hamiltonian. Consequently, the new phase–space variable are constants of the motion, *i.e.*

$$H(q, p) \longrightarrow K(Q, P) \equiv 0 \quad \implies \quad \dot{Q} = \frac{\partial K}{\partial P} \equiv 0, \quad \dot{P} = -\frac{\partial K}{\partial Q} \equiv 0. \quad (1.1)$$

The solution to this problem is given by the Classical Hamilton–Jacobi equation (CHJE). Since the transformations are canonical the phase space variables are taken as independent variables and their functional dependence is only extracted from the solution of the CHJE via the functional relation

$$p = \frac{\partial S(q)}{\partial q}, \quad (1.2)$$

where $S(q)$ is Hamilton's principal function. The fundamental uncertainty relations of quantum mechanics imply that the phase–space variables are not independent. The equivalence postulate of quantum mechanics therefore requires the existence of trivialising coordinate transformations for any physical system, but the phase–space variables

are not independent in the application of the trivialising transformations. They are related by a generating function, via (1.2), which transforms as a scalar function under the transformations. That is,

$$(q, S_0(q), p = \frac{\partial S_0}{\partial q}) \longrightarrow (q^v, S_0^v(q^v), p^v = \frac{\partial S_0^v}{\partial q^v}), \quad (1.3)$$

where $S_0(q)$ is the generating function in the stationary case. It is instrumental to study the stationary case in order to see the symmetry structure that underlies quantum mechanics. The consistency of the equivalence hypothesis implies that the Hamilton–Jacobi equation retains its form under coordinate transformations. However, this cannot be implemented in classical mechanics. The CSHJE for a particle moving under the influence of a velocity independent potential $V(q)$ is given by

$$\frac{1}{2m} \sum_{i=1}^N \left(\frac{\partial S}{\partial q_i} \right)^2 + \mathcal{W}(q) = 0, \quad (1.4)$$

where $\mathcal{W}(q) \equiv V(q) - E$. Under a change of coordinates $q \rightarrow q^v$ we have (by (1.3))

$$\frac{\partial S^v(q^v)}{\partial q_j^v} = \frac{\partial S(q)}{\partial q_j^v} = \sum_i \frac{\partial S(q)}{\partial q_i} \frac{\partial q_i}{\partial q_j^v}, \quad (1.5)$$

which we can write as $\mathbf{p}^v = \mathbf{J}^v \mathbf{p}$, where $J_{ij}^v = \frac{\partial q_i}{\partial q_j^v}$ is the Jacobian matrix connecting the coordinate systems q and q^v , and where, $p_i = \frac{\partial S}{\partial q_i}$. Then

$$\sum_j \left(\frac{\partial S^v}{\partial q_j^v} \right)^2 = |\mathbf{p}^v|^2 = \left(\frac{|\mathbf{p}^v|^2}{|\mathbf{p}|^2} \right) |\mathbf{p}|^2 = (p^v|p) |\mathbf{p}|^2,$$

where we have defined

$$(p^v|p) \equiv \frac{|\mathbf{p}^v|^2}{|\mathbf{p}|^2} = \frac{\mathbf{p}^{v\top} \mathbf{p}^v}{\mathbf{p}^\top \mathbf{p}} = \frac{\mathbf{p}^\top \mathbf{J}^{v\top} \mathbf{J}^v \mathbf{p}}{\mathbf{p}^\top \mathbf{p}}. \quad (1.6)$$

It is seen that the first term in eq. (1.4) transforms as a quadratic differential under the v -map eq. (1.3). Since $S_0^v(q^v)$ must satisfy the CSHJE, covariance of the HJ equation under the v -transformations implies that the second term in eq. (1.4) transforms as a quadratic differential. That is

$$\mathcal{W}^v(q^v) = (p^v|p) \mathcal{W}(q). \quad (1.7)$$

In particular, for the $\mathcal{W}^0(q^0) \equiv 0$ state we have,

$$\mathcal{W}^0(q^0) \longrightarrow \mathcal{W}^v(q^v) = (p^v|p^0) \mathcal{W}^0(q^0) = 0. \quad (1.8)$$

This means that \mathcal{W}^0 is a *fixed point* under v -maps, *i.e.* it cannot be connected to other states. Hence, we conclude that the equivalence postulate cannot be implemented consistently in classical mechanics.

2 The cocycle condition

Consistent implementation of the equivalence postulate necessitates the modification of classical mechanics, which entails adding a yet to be determined function, $\mathcal{Q}(q)$, to the CSHJE. This augmentation produces the Quantum Stationary Hamilton–Jacobi Equation (QSHJE)

$$\frac{1}{2m} \left(\frac{\partial S(q)}{\partial q} \right)^2 + \mathcal{W}(q) + \mathcal{Q}(q) = 0, \quad (2.1)$$

where $\mathcal{W}(q) = V(q) - E$. It is noted that the combination $\mathcal{W}(q) + \mathcal{Q}(q)$ transforms as a quadratic differential under coordinate transformations, whereas each of the functions $\mathcal{W}(q)$ and $\mathcal{Q}(q)$ transforms as a quadratic differential up to an additive term, *i.e.* under $q^a \rightarrow q^v(q)$ we have,

$$\begin{aligned} \mathcal{W}^a(q^a) \rightarrow \mathcal{W}^v(q^v) &= (p^v|p^a) \mathcal{W}^a(q^a) + (q^a; q^v) \\ \mathcal{Q}^a(q^a) \rightarrow \mathcal{Q}^v(q^v) &= (p^v|p^a) \mathcal{Q}^a(q^a) - (q^a; q^v). \end{aligned}$$

and

$$(\mathcal{W}(q^a) + \mathcal{Q}(q^a)) \rightarrow (\mathcal{W}^v(q^v) + \mathcal{Q}^v(q^v)) = (p^v|p^a) (\mathcal{W}^a(q^a) + \mathcal{Q}^a(q^a))$$

All physical states with a non-trivial $\mathcal{W}(q)$ then arise from the inhomogeneous part in the transformation of the trivial state $\mathcal{W}^0(q^0) \equiv 0$, *i.e.* $\mathcal{W}(q) = (q^0; q)$. Considering the transformation $q^a \rightarrow q^b \rightarrow q^c$ versus $q^a \rightarrow q^c$ gives rise to the cocycle condition on the inhomogeneous term

$$(q^a; q^c) = (p^c|p^b) [(q^a; q^b) - (q^c; q^b)]. \quad (2.2)$$

The *cocycle condition* eq. (2.2) embodies the essence of quantum mechanics in the equivalence postulate approach. Furthermore, it reveals the basic symmetry properties that underly quantum mechanics. It is proven [1, 2] that the cocycle condition is invariant under D -dimensional Möbius transformations, which include translations, dilatations, rotations and, most crucially, inversions, or reflections, in the unit sphere. The Möbius transformations are, hence, defined on the compactified space $\hat{\mathbb{R}}^D = \mathbb{R}^D \cup \{\infty\}$. Whereas translations, dilatation and rotations map ∞ to itself, inversions exchange $0 \leftrightarrow \infty$. We argue that energy quantisation and the existence of a fundamental length scale in the formalism, together with the invariance of the cocycle condition eq. (2.2) under the Möbius group $M(\hat{\mathbb{R}}^D)$ of transformations, implies that space is compact. The more general situation may be considered in the decompactification limit.

The cocycle condition fixes the functional form of the quantum potential $\mathcal{Q}(q)$. In one dimension the cocycle condition (2.2) fixes the inhomogeneous term

$$(q^a; q^b) = -\beta \{q^a, q^b\} / 4m,$$

where $\{f, q\} = f'''/f' - 3(f''/f')^2/2$ the Schwarzian derivative and β is a constant with the dimension of an action. In one dimension the Quantum Hamilton–Jacobi equation is given in terms of a basic Schwarzian identity,

$$\left(\frac{\partial S(q)}{\partial q} \right)^2 = \frac{\beta^2}{2} \left(\left\{ e^{\frac{2i}{\beta} S}, q \right\} - \{S, q\} \right) \quad (2.3)$$

Making the identification

$$\mathcal{W}(q) = V(q) - E = -\frac{\beta^2}{4m} \left\{ e^{\frac{i2S_0}{\beta}}, q \right\}, \quad (2.4)$$

and

$$\mathcal{Q}(q) = \frac{\beta^2}{4m} \{S_0, q\}, \quad (2.5)$$

we have that S_0 is the solution of the Quantum Stationary Hamilton–Jacobi equation (QSHJE),

$$\frac{1}{2m} \left(\frac{\partial S_0}{\partial q} \right)^2 + V(q) - E + \frac{\hbar^2}{4m} \{S_0, q\} = 0. \quad (2.6)$$

The Schwarzian identity, eq. (2.3), is generalised in higher dimensions by the basic quadratic identity

$$\alpha^2 (\nabla S_0)^2 = \frac{\Delta(R e^{\alpha S_0})}{R e^{\alpha S_0}} - \frac{\Delta R}{R} - \frac{\alpha}{R^2} \nabla \cdot (R^2 \nabla S_0), \quad (2.7)$$

which holds for any constant α and any functions R and S_0 . Then, if R satisfies the continuity equation $\nabla \cdot (R^2 \nabla S_0) = 0$, and setting $\alpha = i/\hbar$, we have

$$\frac{1}{2m} (\nabla S_0)^2 = -\frac{\hbar^2}{2m} \frac{\Delta(R e^{\frac{i}{\hbar} S_0})}{R e^{\frac{i}{\hbar} S_0}} + \frac{\hbar^2}{2m} \frac{\Delta R}{R}. \quad (2.8)$$

In analogy with the one dimensional case we make identifications,

$$\mathcal{W}(q) = V(q) - E = \frac{\hbar^2}{2m} \frac{\Delta(R e^{\frac{i}{\hbar} S_0})}{R e^{\frac{i}{\hbar} S_0}}, \quad (2.9)$$

$$\mathcal{Q}(q) = -\frac{\hbar^2}{2m} \frac{\Delta R}{R}. \quad (2.10)$$

Eq. (2.9) implies the D -dimensional Schrödinger equation

$$\left[-\frac{\hbar^2}{2m} \Delta + V(q) \right] \Psi = E \Psi. \quad (2.11)$$

and the general solution

$$\Psi = R(q) \left(A e^{\frac{i}{\hbar} S_0} + B e^{-\frac{i}{\hbar} S_0} \right). \quad (2.12)$$

We note that consistency of the equivalence postulate formalism necessitates that the two solutions of the second order Schrödinger equation are retained. This is reminiscent of relativistic quantum mechanics in which both the positive and negative energy solutions are retained. We can replace the gradient in eq. (2.7) by a four vector derivative in Minkowski space. This produces the generalisation of the formalism to the relativistic case and the Schrödinger equation, eq. (2.11), is replaced by the Klein–Gordon equation. The time-dependent Schrödinger equation arises in the limit $c \rightarrow \infty$. Similarly, the cocycle condition eq. (2.2) generalises to Minkowski space by replacing the Euclidean metric with the Minkowski metric. It is important to emphasize that the equivalence postulate approach to quantum mechanics does not represent a modification or interpretation

of quantum mechanics but its derivation from a basic geometrical principle. As such it reveals the geometrical structures underlying quantum mechanics and in that respect provides an intrinsic framework to explore the quantum space–time. It is further noted that the cocycle condition, eq. (2.2), is completely universal. Hence, its generalisation to curved space provides a background independent approach to quantum gravity. In this respect the equivalence postulate approach reveals the interplay between quantum variables, encoded $R(q)$ and $S(q)$, versus the classical background parameters. For example, in ref. [4] we showed that the QHJE does not admit a consistent time parameterisation of quantum trajectories. In this respect, therefore, time cannot be defined as a quantum observable, but is merely a classical background parameter. Generalising this observation to relativistic space–time entails that space–time cannot be consistently defined as a quantum observable. Instead, the quantum data is encoded in the cocycle condition and the corresponding quadric identity in the relevant domain, *i.e.* in curved space–time. In this respect, we note that the inhomogeneous term can be written in the general form [2],

$$(q^a; q^b) = (p^b|p^a)\mathcal{Q}^a(q^a) - \mathcal{Q}^b(q^b) = -\frac{\hbar^2}{2m} \left[(p^b|p^a) \frac{\Delta^a R^a}{R^a} - \frac{\Delta^b R^b}{R^b} \right], \quad (2.13)$$

which shows how the information on the inhomogeneous term is encoded in the functions $R(q)$ and $S(q)$.

3 The quantum closet

The invariance of the cocycle condition under Möbius transformations implies that space is compact. Let us gather the evidence for this claim. In the one dimensional case we see from eq. (2.9) that the QSHJE is equivalent to the equation $\{w, q\} = -4m(V(q) - E)/\hbar^2$ where w is the ratio of the two solutions of the Schrödinger equation. It follows from the Möbius invariance of the cocycle condition that $w \neq \text{const}$, $w \in C^2(\hat{\mathbb{R}})$ with w'' differentiable on \mathbb{R} , where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and

$$w(-\infty) = \begin{cases} +w(+\infty) & \text{if } w(-\infty) \neq \pm\infty, \\ -w(+\infty) & \text{if } w(-\infty) = \pm\infty. \end{cases} \quad (3.1)$$

Furthermore, denoting by q_- (q_+) the lowest (highest) q for which $V(q) - E$ changes sign, we prove the general theorem [1],

If

$$V(q) - E \geq \begin{cases} P_-^2 > 0, & q < q_-, \\ P_+^2 > 0, & q > q_+, \end{cases} \quad (3.2)$$

then $w = \psi^D/\psi$ is a local self-homeomorphism of $\hat{\mathbb{R}}$ iff the Schrödinger equation has an $L^2(\mathbb{R})$ solution.

Since the QSHJE is defined if and only if w is a local self-homeomorphism of $\hat{\mathbb{R}}$, this theorem implies that energy quantisation *directly* follows from the geometrical gluing conditions of w at $q = \pm\infty$, as implied by the equivalence postulate, which in turn imply that the one dimensional space is compact. In turn the compactness of space implies that the energy of the free quantum particle is quantised and that time parameterisation of trajectories is ill defined either via Bohm–de Broglie mechanical definition, or via Floyd’s

definition by using Jacobi's theorem [3]. The Möbius invariance of the cocycle condition in D dimensions then implies that the D dimensional space is compact.

Generalisation of the cocycle condition to curved space suggests a background independent approach to quantum gravity. The connection with gravity and with an internal structure of elementary particles is implied due to the existence of an intrinsic fundamental length scale in the formalism, and the association of the quantum potential, $\mathcal{Q}(q)$, with a curvature term [1, 5, 6]. To see the origin of that we can again examine the stationary one dimensional case with $\mathcal{W}^0(q^0) \equiv 0$. In this case the Schrödinger equation takes the form

$$\frac{\partial^2}{\partial q^2} \psi = 0,$$

with the two linearly independent solutions being $\psi^D = q^0$ and $\psi = \text{const.}$ Consistency of the equivalence postulate dictates that both solutions of the Schrödinger equation must be retained. The solution of the corresponding QHJE is given by [1]

$$e^{\frac{2i}{\hbar} S_0^0} = e^{i\alpha} \frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0},$$

where ℓ_0 is a constant with the dimension of length [1], and the conjugate momentum $p_0 = \partial_{q^0} S_0^0$ takes the form

$$p_0 = \pm \frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2}. \quad (3.3)$$

It is seen that p_0 vanishes only for $q^0 \rightarrow \pm\infty$. The requirement that in the classical limit $\lim_{\hbar \rightarrow 0} p_0 = 0$ implies that we can set [1]

$$\text{Re } \ell_0 = \lambda_p = \sqrt{\frac{\hbar G}{c^3}}, \quad (3.4)$$

i.e. we identify $\text{Re } \ell_0$ with the Planck length. The interpretation of the quantum potential as a curvature term [1, 6] implies that elementary particles possess an internal structure, *i.e.* points do not have curvatures. This suggests possible connection with theories of extended objects.

If the universe is compact it would imply the existence of an intrinsic energy scale reminiscent of the Casimir effect. Taking the present size of the observable universe would imply a very small energy scale, which is essentially unobservable [6]. However, given the indication of a larger energy scale in the Cosmic Microwave Background (CMB) Radiation suggests the possibility of observing the imprints of compactness of the universe in the CMB in the current [7] or future CMB observatories. Indeed, the possibility of signatures of a non-trivial topology in the CMB has been of recent interest [8]. Additional experimental evidence for the equivalence postulate approach to quantum mechanics may arise from modifications of the relativistic energy-momentum relation [9], which affects the propagation of light from gamma ray bursts [10].

4 The decompactification limit

The Möbius invariance of cocycle condition may only be implemented if space is compact. We may contemplate that the decompactification limit represents the case when the spectrum of the free quantum particle becomes continuous. In that case time parameterisation

of quantum trajectories is consistent with the definition of time by using Jacobi's theorem [3, 1, 4]. However, I argue that the decompactification limit in fact coincides with the classical limit. To see that this may be the case we examine again the case of the free particle in one dimension. The quantum potential associated with the state $W^0 \equiv 0$ is given by

$$\mathcal{Q}^0 = \frac{\hbar^2}{4m} \{S_0^0, q^0\} = -\frac{\hbar^2 (\text{Re } \ell_0)^2}{2m} \frac{1}{|q^0 - i\ell_0|^4}. \quad (4.1)$$

We note that the limit $q^0 \rightarrow \infty$ coincides with the limit $\mathcal{Q}^0 \rightarrow 0$, *i.e.* with the classical limit.

5 Conclusions

Heisenberg's uncertainty principle mandates that the phase-space variables cannot be treated as independent variables. The classical Hamilton–Jacobi trivialising transformations are in direct conflict with this fact. Reconciling the Hamilton–Jacobi theory to quantum mechanics leads to the quantum Hamilton–Jacobi equation (QHJE). In turn, the QHJE implies a basic cocycle condition that underlies quantum mechanics. The cocycle condition holds in any background and provides a framework for the background independent formulation of quantum gravity. The cocycle condition is invariant under D -dimensional finite Möbius transformations with respect to the Euclidean or Minkowski metrics. Its invariance under D -dimensional Möbius transformations implies that space is compact, which may have an imprint in the cosmic microwave background radiation.

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